Finsleroid–Finsler Space with Berwald and Landsberg Conditions

G.S. Asanov

Division of Theoretical Physics, Moscow State University
119992 Moscow, Russia
(*e-mail: asanov@newmail.ru*)

Abstract

We formulate the notion of the Finsleroid–Finsler space, including the positive–definite as well as indefinite cases. The associated concepts of angle, scalar product, and the distance function are elucidated. If the Finsleroid–Finsler space is of Landsberg type, then the Finsleroid charge is a constant. The Finsleroid–Finsler space proves to be a Berwald space if and only if the Finsleroid–axis 1-form is parallel with respect to the associated Riemannian metric and, simultaneously, the Finsleroid charge is a constant. The necessary and sufficient conditions for the Finsleroid–Finsler space to be of the Landsberg type are found, which are explicit and simple. The structure of the associated curvature tensors has been elucidated.

Key words: Finsler geometry, metric spaces, angle, scalar product.

1. Introduction and synopsis of conclusions

The ideas and methods of the Riemannian geometry admit ingenious success-ful metric extensions by using various Finslerian metric functions (see [1-12]). The Finsleroid–type metric function K, which properties have been systematically exposed in the previous work [12–14], is interesting because of entailing numerous remarkable implications, including the positive–definiteness, the constant curvature of the indicatrix, the special algebraic form of the Cartan tensor, the occurrence of the angle and scalar product, the particular conformal properties, the distance as well as the explicit solu-tions to the geodesic equations in the respective Finsleroid–Minkowski space, *etc.* The Finsleroid is the unit ball defined by the function K under treating K as a Minkowskian norm. The respective Finslerian indicatrix is the boundary of the Finsleroid, which is strongly convex and rotund.

Below we introduce an appropriate extension to work on curved manifolds, such that the Finsleroid–Minkowski structure gets placed on each tangent space. Namely, given a Riemannian space under assumption that the space admits a 1-form, we use the involved Riemannian metric $\mathcal{S}$ and the 1-form, introducing also the scalar g which plays the role of the Finsleroid charge, to construct on the Riemannian space the Finsleroid–Finsler metric function K such that in each associated tangent space the vector entering the 1-form plays the role of the axis of the Finsleroid. We call the result the *Finsleroid–Finsler Space*, the $\mathcal{FF}_g^{PD}$–*space* for short. The space thus appeared is actually some structure over a Riemannian space. Whenever $g = 0$, the Finsleroid is the Euclidean unit ball and whence the $\mathcal{FF}_g^{PD}$–space is a Riemannian space.

Clarifying and studying the entailed Finslerian geometric structure, the connection and curvature included, seems to be an urgent task. First of all, it is of interesting to clarify the Berwald and Landsberg types among the metric functions of the $\mathcal{FF}_g^{PD}$–space, for the types play an important role in the modern Finsler geometry (see [7-9,15]) and sound intriguing to apply in developments of physical applications. Accordingly, we start with raising up the following questions.

QUESTION 1. *Do there exist non–trivial conditions under which the Finsleroid–Finsler space is a Berwald space?*

QUESTION 2. *Can the Finsleroid–Finsler space be of Landsberg type and not of Berwald type?*

Let $\{A_{ijk}\}$ denote the Cartan tensor and $A_i := g^{jk}A_{ijk}$ be the contraction thereof by means of the Finslerian metric tensor. Among the conditions

$$A_{jmn|i} = 0, \tag{1.1}$$

$$A_{j|i} = 0, \tag{1.2}$$

$$\dot{A}_{jmn} = 0, \tag{1.3}$$

$$\dot{A}_j = 0 \tag{1.4}$$

the last one is obviously weakest. We shall comply with the notation adopted in the

book [7], so that the bar means the h-covariant derivative, the dot over A stands for the action of the operator $|il^i$, where $l^i = y^i/K$, and the indices $i, j, \ldots$ refer to natural local coordinates (denoted by x^i); the notation y is used for tangent vectors;

$$\dot{X} = X_{|i}l^i \tag{1.5}$$

for any tensor X. For purposes of calculations performed below, the smoothness of the class C^3 is sufficient for the associated Riemannian space $\mathcal{R}_N$ as well as for the input 1-form $b = b_i(x)y^i$ and the scalar $g(x)$. The upperscript "PD" emphasizes the Positive–Definiteness (see the determinant value (2.25) in Section 2) of the $\mathcal{FF}_g^{PD}$–space under study.

A Finsler space is said to be a *Berwald space* if the condition (1.1) holds. A *Landsberg space* under the Finslerian consideration is characterized by the condition (1.3).

INITIAL OBSERVATIONS. *If the weakly Landsberg condition* (1.4) *holds, then the Finsleroid charge is independent of points* x *of the background manifold:*

$$g = \text{constant.} \tag{1.6}$$

The weakly Landsberg condition (1.4) *entails the Landsberg condition* (1.3), *and the weakly Berwald condition* (1.2) *entails the Berwald condition* (1.1).

A mere glance at the algebraic structure (2.27)–(2.28) of the $\mathcal{FF}_g^{PD}$–space Cartan tensor is sufficient to recognize the validity of these assertions.

Below, we shall confine the treatment to the dimensions

$$N \geq 3, \tag{1.7}$$

keeping in mind that the two–dimensional $\mathcal{FF}_g^{PD}$–space is of simple structure (we shall present required notes in the last section Conclusions).

In our consideration, an important role will be played by the *lengthened angular metric tensor*

$$\mathcal{H}_{ij} := h_{ij} - \frac{A_i A_j}{A_n A^n} \tag{1.8}$$

which fulfills the identities

$$\mathcal{H}_{ij}y^j = 0, \qquad \mathcal{H}_{ij}A^j = 0, \qquad \mathcal{H}_{ij}b^j = 0; \tag{1.9}$$

h_{ij} is the traditional Finslerian angular metric tensor (see (2.24)).

We set forth the following theorems.

Theorem 1. *Whenever the Finsleroid charge is a constant, the condition* (1.6), *the representation*

$$\dot{A}_i = \frac{Ng}{2q}\mathcal{H}_i{}^m P_m \tag{1.10}$$

takes place, where

$$P_m = y^j \nabla_j b_m + \frac{1}{2}gqb^j\nabla_j b_m. \tag{1.11}$$

Here, q is the quantity (2.3) and ∇ stands for the covariant derivative with respect to the underlined Riemannian metric $\mathcal{S}$.

Whenever the Finsleroid charge is a constant, all the three identities

$$\dot{A}_j y^j = 0, \qquad \dot{A}_j A^j = 0, \qquad \dot{A}_j b^j = 0 \tag{1.12}$$

are fulfilled. Indeed, the first identity is universal for any Finsler space. In the $\mathcal{FF}_g^{PD}-$ space, the contraction $A^h A_h$ is a function of the Finsleroid charge g (see (2.28)), so that whenever $g = const$ we must have $\dot{A}_j A^j = 0$. The third identity is a direct implications of the first and second ones because A_i are linear combinations of y_i and b_i (see (A.7) in Appendix A).

Comparing (1.12) with (1.9) gives a handy clue to search for the object $\dot{A}_i$ of the $\mathcal{FF}_g^{PD}$–space to be a contraction of the tensor $\mathcal{H}^k{}_i$ by a vector. The straightforward and attentive (and rather lengthy) calculations result in (1.10) and (1.11)); the vanishing

$$b^j \nabla_i b_j = 0 \tag{1.13}$$

(the vector b_i is of the unit Riemannian length according to the formula (2.5) of the next section) nullifies many terms appearing while calculating. In particular, applying (1.13) to (1.11) results in

$$b^i P_i = 0. \tag{1.14}$$

Let us turn to the Berwald condition.

Theorem 2. *The $\mathcal{FF}_g^{PD}$–space is a Berwald space if and only if in addition to the constancy (1.6) of the Finsleroid charge the Finsleroid–axis field $b_i(x)$ is parallel, $\nabla_i b_j = 0$, with respect to the input Riemannian metric $\mathcal{S}$.*

Under the conditions set forth in this theorem, the hh-connection and the hh-curvature tensor are the Riemannian connection and the Riemannian curvature tensor produced by the input Riemannian metric $\mathcal{S}$, and the hv-curvature tensor P vanishes identically; at the same time, the basic Finslerian metric tensor remains being non–Riemannian (unless $g = 0$).

Theorem 2 is a particular case of the following.

Theorem 3. *The $\mathcal{FF}_g^{PD}$–space is a Landsberg space if and only if in addition to the constancy (1.6) of the Finsleroid charge the Finsleroid–axis 1-form $b = b_i(x)y^i$ is closed*

$$\nabla_j b_i - \nabla_i b_j = 0 \tag{1.15}$$

and obeying the condition

$$\nabla_i b_j = k(a_{ij} - b_i b_j) \tag{1.16}$$

with a scalar $k = k(x)$.

In such a space, it proves possible to obtain a simple explicit representation for each distinguished Finslerian tensor. In particular, we have the following.

Theorem 4. *In any Landsberg case of the $\mathcal{FF}_g^{PD}$–space, the tensor*

$$P_{ijkl} := -A_{ijk|l} \tag{1.17}$$

is of the special algebraic form

$$P_{ijkl} = -\frac{gB}{2qK}\big(\mathcal{H}_{ij}\mathcal{H}_{kl} + \mathcal{H}_{ik}\mathcal{H}_{jl} + \mathcal{H}_{jk}\mathcal{H}_{il}\big)k, \tag{1.18}$$

and therefore,

$$A_{i|j} = \frac{NgB}{2qK}\mathcal{H}_{ij}k. \tag{1.19}$$

In (1.16), (1.18), and (1.19), the case $k = 0$ corresponds to the Berwlad type.

In the above formulae, K is the Finsleroid metric function given by (2.14)–(2.18) and B is the characteristic quadratic form (2.11). The formula (1.18) is remarkable in that the tensor (1.17) is lucidly constructed in terms of the lengthened angular metric tensor $\mathcal{H}_{ij}$ defined by (1.8). Complete representation for the hv-curvature tensor $P_k{}^i{}_{mn}$ can be found in the end of Appendix A below.

The tensor P_{ijkl} given by (1.18) is *totally symmetric* in all four of its indices and, owing to (1.9), is entailing the identities

$$P_{ijkl}y^l = 0, \qquad P_{ijkl}A^l = 0, \qquad P_{ijkl}b^l = 0, \tag{1.20}$$

which in turn entail

$$A_{i|j}y^j = 0, \qquad A_{i|j}A^j = 0, \qquad A_{i|j}b^j = 0. \tag{1.21}$$

The sufficiency in Theorem 3 is easy to verify by direct (two-night-long!) computations. As regards the necessity, the representation (1.10)–(1.11) given by Theorem 1 proves to be not only elegant but also handy in permitting the equation $\dot{A}_i = 0$ to be resolved, using a rather simple Lemma proven in Section 3. The shortest way to arrive at the representation (1.18) stated by Theorem 4 is to apply the Finslerian formula (3.1) of Section 3.

The Landsberg–characteristic condition (1.16) entails the equality

$$y^i y^j \nabla_j b_i = kq^2 \tag{1.22}$$

and also the identity

$$b^j \nabla_j b_i = 0 \tag{1.23}$$

(supplementary to (1.13)). If we consider the *b-lines* which comprise the congruence tangent to the vector field $b^i(x)$, noting also the Riemannian unity (see (2.5) below in Section 2) for the length of the vectors b^i, we are justified in concluding from (1.23) that the following proposition is a truth.

Proposition 1. *In any Landsberg–case of the $\mathcal{FF}_g^{PD}$–space, the b-lines are geodesics of the associated Riemannian space.*

When the Landsberg case takes place for a $\mathcal{FF}_g^{PD}$–space, the computation of the key contraction $\gamma^i{}_{nm}y^n y^m$, with $\gamma^i{}_{nm}$ being the associated Finslerian Christoffel symbols, leads us to the following astonishingly simple result:

$$\gamma^i{}_{nm}y^n y^m = gqk(y^i - bb^i) + a^i{}_{nm}y^n y^m, \tag{1.24}$$

where $a^i{}_{km}$ are the Christoffel symbols constructed from the input Riemannian metric $\mathcal{S}$.

Since the representation (1.24) entails the nullification

$$b_i(\gamma^i{}_{nm}y^n y^m - a^i{}_{nm}y^n y^m) = 0 \tag{1.25}$$

(notice that $b^i b_i = 1$ in accordance with the formulas (2.5) and (2.8) of the next section), we are justified in stating the following remarkable assertion.

Proposition 2. *In any Landsberg case of the $\mathcal{FF}_g^{PD}$–space, the covariant derivatives of the vector field $b_i(x)$ with respect to the Finsler connection and with respect to the associated Riemannian connection are equal to one another:*

$$b_{i|j} = \nabla_j b_i. \tag{1.26}$$

The commutator

$$(\nabla_m \nabla_n - \nabla_n \nabla_m) b_k = \widetilde{k}_m (a_{nk} - b_n b_k) - \widetilde{k}_n (a_{mk} - b_m b_k) \tag{1.27}$$

(insert the Landsberg–characteristic condition (1.16) in the left–hand part) is valid with the vector

$$\widetilde{k}_n = \frac{\partial k}{\partial x^n} + k^2 b_n. \tag{1.28}$$

Finally, the formulas obtained in the end of Appendix A enable us to formulate the following.

Proposition 3. *In any Landsberg case of the $\mathcal{FF}_g^{PD}$–space, the tensor*

$$\rho_{ij} := \frac{1}{2}(R_i{}^m{}_{mj} + R^m{}_{ijm}) - \frac{1}{2}g_{ij}R^{mn}{}_{nm} \tag{1.29}$$

is covariantly conserved:

$$\rho^i{}_{j|i} \equiv 0. \tag{1.30}$$

Lengthy straightforward calculations lead to the following assertion.

Proposition 4. *Let us be given a $\mathcal{FF}_g^{PD}$–space with constant Finsleroid charge. Then the tensor $\dot{A}_{ikl}$ is of the following special algebraic structure:*

$$\dot{A}_{ikl} = \frac{g}{2q}\Big(c_i \mathcal{H}_{kl} + c_k \mathcal{H}_{li} + c_l \mathcal{H}_{ik}\Big) \tag{1.31}$$

with

$$c_i = \Big(y^j \nabla_j b_i + \frac{1}{2}gqb^j \nabla_j b_i\Big) - \frac{1}{q^2}\Big(y^j y^h \nabla_j b_h + \frac{1}{2}gqy^h b^j \nabla_j b_h\Big)v_i, \tag{1.32}$$

where $v_i = a_{in}y^n - bb_i$.

This just entails the representation

$$\dot{A}_i = \frac{gN}{2q}c_i \tag{1.33}$$

which is equivalent to (1.10). The vector (1.32) fulfills the identities

$$c_i y^i = 0, \qquad c_i b^i = 0, \qquad c_i A^i = 0 \tag{1.34}$$

(notice that $v_i b^i = 0$).

Below, in Section 2 we formulate rigorously the basic concepts which underline the notion of the $\mathcal{FF}_g^{PD}$–space. The Finsleroid–Finsler metric function, K, given by the explicit formulas (2.14)–(2.18), is defined by the triple: a Riemannian metric S, a 1-form b, and a scalar $g(x)$. We attribute to the vector $b^i(x)$ of the 1-form the geometrical

meaning of the direction of the *axis* of the Finsleroid supported by a point $x \in M$, and to the scalar $g(x)$ the meaning of "the geometric charge", particularly "the Finsleroid charge". Would the charge $g(x)$ be zero, the function K reduce to the Riemannian S. The special and lucid algebraic form (2.27)–(2.28) of the associated Cartan tensor gives rise to many essential simplifications, including the constant curvature of the Finsleroid indicatrix. The explicit formula (2.34) is proposed for the angle that equals the length of a piece of the Finslerian unit circle on the Finsleroid indicatrix (the meaning of such an angle complies totally with the ordinary meaning of the Euclidean "induced angle" on the tangent spheres in the Riemannian geometry). Having obtained the angle, we obtain the scalar product, too.

In Section 3 we place the arguments which verify the validity of the above theorems.

In Section 4 the structure of the associated curvature tensor is elucidated in various aspects.

In Section 5 we open up the remarkable phenomenon that the Landsberg–type $\mathcal{FF}_g^{PD}$–space is underlined by a warped product metric of the associated Riemannian space with respect to the b-geodesic coordinates.

In Section 6 we indicate a convenient possibility to re–formulate the theory in the indefinite case of the time–space type. The resultant space will be referred to as the $\mathcal{FF}_g^{SR}$–space. Although the above formulas and the analysis presented in Sections 3–5 are all referred to the positive–definite case, a simple additional consideration (omitted in the present paper) shows that all the principal representations and conclusions can straightforwardly be re–addressed to the indefinite pseudo–Finsleroid space exposed in Section 6: to do this, in most cases it is sufficient merely to change in formulas the sign of the Finsleroid charge g. Whence all Theorems 1-7 and Propositions 1–5 remain valid under transition from the positive–definite Finsleroid–Finsler $\mathcal{FF}_g^{PD}$–space to the indefinite pseudo–Finsleroid–Finsler $\mathcal{FF}_g^{SR}$–space.

The basic geometric ideas involved and possible applied potentialities of the spaces proposed are emphasized in Conclusions.

Appendix A summarizes the important representations which support the calculations involved.

2. Positive–Definite Case

Let M be an N-dimensional differentiable manifold. Suppose we are given on M a Riemannian metric $\mathcal{S} = S(x, y)$, where $x \in M$ denotes points and $y \in T_x M$ means tangent vectors. Denote by $\mathcal{R}_N = (M, \mathcal{S})$ the obtained N-dimensional Riemannian space.

Let us assume that the manifold M admits a non–vanishing 1-form, to be denoted as $\beta = \beta(x, y)$, and introduce the associated normalized 1-form $b = b(x, y)$ such that the Riemannian length of the involved vector be equal to 1. With respect to natural local coordinates in the space $\mathcal{R}_N$ we have the local representations

$$b = b_i(x)y^i, \tag{2.1}$$

$$S = \sqrt{a_{ij}(x)y^i y^j}, \tag{2.2}$$

$$q = \sqrt{r_{ij}(x)y^i y^j} \tag{2.3}$$

with

$$r_{ij}(x) = a_{ij}(x) - b_i(x)b_j(x) \tag{2.4}$$

and

$$a^{ij}b_i b_j = 1. \tag{2.5}$$

The decomposition

$$S^2 = b^2 + q^2 \tag{2.6}$$

introduces a Riemannian metric q on the $(N-1)$–dimensional Riemannian space $\mathcal{R}_{N-1} \subset \mathcal{R}_N$ given rise to by the orthogonality relative to the 1-form b, for from (2.4) and (2.5) it follows that

$$b^i r_{ij} = 0, \tag{2.7}$$

where

$$b^i := a^{ik} b_k. \tag{2.8}$$

Finally, we introduce on M a scalar field $g = g(x)$ subject to ranging

$$-2 < g(x) < 2, \tag{2.9}$$

and apply the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \qquad G = g/h. \tag{2.10}$$

The *characteristic quadratic form*

$$B(x,y) := b^2 + gqb + q^2 \equiv \frac{1}{2}\Big[(b + g_+ q)^2 + (b + g_- q)^2\Big] > 0 \tag{2.11}$$

where $g_+ = \dfrac{1}{2}g + h$ and $g_- = \dfrac{1}{2}g - h$, is of the negative discriminant

$$D_{\{B\}} = -4h^2 < 0 \tag{2.12}$$

and, therefore, is positively definite. In the limit $g \to 0$, the definition (2.11) degenerates to the quadratic form (2.6) of the input Riemannian metric tensor:

$$B|_{g=0} = S^2. \tag{2.13}$$

In terms of these concepts, we extend the notion of the Finsleroid metric function K proposed early in the framework of the Minkowski space (see [12–14]) and introduce the following definition adaptable to consideration on manifolds.

Definition. The scalar function $K(x,y)$ given by the formulae

$$K(x,y) = \sqrt{B(x,y)}\, J(x,y) \tag{2.14}$$

and

$$J(x,y) = e^{\frac{1}{2}G\Phi(x,y)}, \tag{2.15}$$

where

$$\Phi(x,y) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{L(x,y)}{hb}\Big), \qquad \text{if} \quad b \geq 0, \tag{2.16}$$

and

$$\Phi(x, y) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\left(\frac{L(x, y)}{hb}\right), \qquad \text{if} \quad b \leq 0, \tag{2.17}$$

with

$$L(x, y) = q + \frac{g}{2}b, \tag{2.18}$$

is called the *Finsleroid–Finsler metric function.*

The positive (not absolute) homogeneity holds fine: $K(x, \lambda y) = \lambda K(x, y)$ for all $\lambda > 0$.

Sometimes it is convenient to use also the function

$$A(x, y) = b + \frac{g}{2}q. \tag{2.19}$$

We have

$$L^2 + h^2 b^2 = B, \qquad A^2 + h^2 q^2 = B. \tag{2.20}$$

Definition. The arisen space

$$\mathcal{FF}_g^{PD} := \{\mathcal{R}_N;\ b(x, y);\ g(x);\ K(x, y)\} \tag{2.21}$$

is called the *Finsleroid–Finsler space.*

Definition. The space $\mathcal{R}_N$ entering the above definition is called the *associated Riemannian space.*

Definition. Within any point $T_x M$, the Finsleroid–metric function $K(x, y)$ produces the *Finsleroid*

$$\mathcal{F}_{g\,\{x\}}^{PD} := \{y \in \mathcal{F}_{g\,\{x\}}^{PD} : y \in T_x M, K(x, y) \leq 1\}. \tag{2.22}$$

Definition. The *Finsleroid Indicatrix* $I_{g\,\{x\}}^{PD} \in T_x M$ is the boundary of the Finsleroid:

$$I_{g\,\{x\}}^{PD} := \{y \in I_{g\,\{x\}}^{PD} : y \in T_x M, K(x, y) = 1\}. \tag{2.23}$$

Since at $g = 0$ the $\mathcal{FF}_g^{PD}$–space is Riemannian, then the body $\mathcal{F}_{g=0\,\{x\}}^{PD}$ is a unit ball and $I_{g=0\,\{x\}}^{PD}$ is a unit sphere.

Definition. The scalar $g(x)$ is called the *Finsleroid charge.* The 1-form b is called the *Finsleroid–axis 1-form.*

Under these conditions, we can explicitly calculate from the function K the distinguished Finslerian tensors, and first of all the covariant tangent vector $\hat{y} = \{y_i\}$, the Finslerian metric tensor $\{g_{ij}\}$ together with the contravariant tensor $\{g^{ij}\}$ defined by the reciprocity conditions $g_{ij}g^{jk} = \delta_i^k$, and the angular metric tensor $\{h_{ij}\}$, by making use of the following conventional Finslerian rules in succession:

$$y_i := \frac{1}{2}\frac{\partial K^2}{\partial y^i}, \qquad g_{ij} := \frac{1}{2}\frac{\partial^2 K^2}{\partial y^i \partial y^j} = \frac{\partial y_i}{\partial y^j}, \qquad h_{ij} := g_{ij} - y_i y_j \frac{1}{K^2}. \tag{2.24}$$

Calculations show that the determinant of the associated Finsleroid metric tensor is everywhere positive:

$$\det(g_{ij}) = \left(\frac{K^2}{B}\right)^N \det(a_{ij}) > 0. \tag{2.25}$$

After that, we can elucidate the algebraic structure of the associated Cartan tensor

$$A_{ijk} := \frac{K}{2}\frac{\partial g_{ij}}{\partial y^k}, \tag{2.26}$$

which leads to the following simple and remarkable result: *The Cartan tensor associated with the Finsleroid–Finsler metric function K is of the following special algebraic form:*

$$A_{ijk} = \frac{1}{N}\left(h_{ij}A_k + h_{ik}A_j + h_{jk}A_i - \frac{1}{A_hA^h}A_iA_jA_k\right) \tag{2.27}$$

with

$$A_hA^h = \frac{N^2}{4}g^2, \tag{2.28}$$

which is in fact a mere adaptation of the Cartan tensor structure known in the Finsleroid–Minkowski approach developed in [12–14]. Elucidating the respective tensor

$$\hat{R}_i{}^j{}_{mn} := \frac{1}{K^2}(A_h{}^j{}_m A_i{}^h{}_n - A_h{}^j{}_n A_i{}^h{}_m) \tag{2.29}$$

describing the curvature of the indicatrix results, upon using (2.27), in the simple representation

$$\hat{R}_{ijmn} = -\frac{A_hA^h}{N^2}(h_{im}h_{jn} - h_{in}h_{jm}).$$

Inserting here (2.28), we are led to the following remarkable assertion. *The indicatrix curvature tensor* (2.29) *of the space $\mathcal{FF}_g^{PD}$ is of the special structure such that*

$$\hat{R}_{ijmn} = S^*(h_{im}h_{jn} - h_{in}h_{jm})/K^2 \tag{2.30}$$

with

$$S^* = -\frac{1}{4}g^2. \tag{2.31}$$

Recalling the known formula $\mathcal{R} = 1 + S^*$ for the indicatrix curvature (see Section 1.2 in [1]), from (2.29) and (2.30) we conclude that

$$\mathcal{R}_{\text{Finsleroid Indicatrix}} = h^2 \equiv 1 - \frac{1}{4}g^2, \tag{2.32}$$

so that

$$0 < \mathcal{R}_{\text{Finsleroid Indicatrix}} \leq 1$$

and

$$\mathcal{R}_{\text{Finsleroid Indicatrix}} \stackrel{g\to 0}{\Longrightarrow} \mathcal{R}_{\text{Euclidean Sphere}} = 1.$$

Geometrically, the fact that the quantity (2.32) is independent of vectors y means: *The Finsleroid indicatrix is a space of constant positive curvature.*

Given any two nonzero tangent vectors $y_1, y_2 \in T_xM$ of a fixed tangent space, we can, by following the previous work [12–14], obtain the $\mathcal{FF}_g^{PD}$–*scalar product*

$$< y_1, y_2 >_{\{x\}} := K(x, y_1)K(x, y_2)\cos\left(\alpha_{\{x\}}(y_1, y_2)\right) \tag{2.33}$$

and the $\mathcal{FF}_g^{PD}$–angle

$$\alpha_{\{x\}}(y_1, y_2) \; := \; \frac{1}{h} \arccos \frac{A(x, y_1)A(x, y_2) + h^2 < y_1, y_2 >_{\{x\}}^{\{r\}}}{\sqrt{B(x, y_1)} \sqrt{B(x, y_2)}}, \tag{2.34}$$

where $< y_1, y_2 >_{\{x\}}^{\{r\}} = r_{ij}(x)y_1^i y_2^j$. The $\mathcal{FF}_g^{PD}$–distance Δs between ends of the vectors in the tangent space is given by the formula

$$(\Delta s)^2 = (K(x, y_1))^2 + (K(x, y_2))^2 - 2K(x, y_1)K(x, y_2) \cos\left(\alpha_{\{x\}}(y_1, y_2)\right), \tag{2.35}$$

which extends the ordinary Euclidean cosine theorem. At equal vectors, the reduction

$$< y, y >_{\{x\}} = (K(x, y))^2 \tag{2.36}$$

takes place, that is, the two-vector scalar product (2.33) reduces exactly to the squared Finsleroid–Finsler metric function.

These scalar product and angle obviously exhibit the symmetry:

$$(y_1, y_2)_{\{x\}} = (y_2, y_1)_{\{x\}}, \qquad \alpha_{\{x\}}(y_1, y_2) = \alpha_{\{x\}}(y_2, y_1). \tag{2.37}$$

They are entirely intermediary, supporting by a point $x \in M$ of the base manifold M (in just the same sense as in the Riemannian geometry) and being independent of any vector element of support. The angle (2.34) furnishes actually the ordinary meaning of the arc–length s cut off by a geodesic piece on the indicatrix:

$$\alpha_{\{x\}}(y_1, y_2) = s_{\{\text{Finsleroid Indicatrix}\}}(x; y_1, y_2). \tag{2.38}$$

Namely, considering the indicatrix together with the geodesic curves thereon, we can calculate the length of the geodesic piece which joins the ends of the unit vectors $l_1 = y_1/K(x, y_1)$ and $l_1 = y_2/K(x, y_2)$, thereby obtaining the right–hand part in (2.34). Taken (2.38) to be the definition for the angle, the explicit representation (2.34) can be verified by direct calculation of the arc length over the Finsleroid indicatrix; the same angle is derivable by postulating the cosine theorem (2.35) (see the previous work [12–14]). The formula (2.34) assigns the angle to be a length of a piece of the Finslerian unit circle on the Finsleroid indicatrix. Whenever the Finsleroid charge is zero, $g = 0$, the Finslerian angle (2.34) reduces to the ordinary Riemannian angle $\arccos\Big[\big(b(x, y_1)b(x, y_2) + < y_1, y_2 >_{\{x\}}^{\{r\}}\big)\Big($ $\big(S(x, y_1)S(x, y_2)\big)^{-1}\Big] = \arccos\Big[a_{ij}(x)y_1^i y_2^j \big(S(x, y_1)S(x, y_2)\big)^{-1}\Big]$.

In particular, if we consider the angle $\alpha_{\{x\}}(y)$ formed by a vector $y \in T_xM$ with the Finsleroid axis in a fixed T_xM, from (2.33) we get the respective value to be

$$\alpha_{\{x\}}(y) \; := \; \frac{1}{h} \arccos \frac{A(x, y)}{\sqrt{B(x, y)}}. \tag{2.39}$$

3. Search for Berwald and Landsberg types of $\mathcal{FF}_g^{PD}$–space

In any Finsler space subjected to the Landsberg–type condition the equality

$$A_{ijk|l} = -\left(F \frac{\partial A_{ijk}}{\partial y^l}\right)_{|s} l^s \tag{3.1}$$

(see (12.4.1) on p. 326 of the book [7]) holds. In the $\mathcal{FF}_g^{PD}$–space under study, the representations (A.23) and (A.24) may be used to just conclude from (3.1) that

$$\mathcal{A}_{ijk|l} = \frac{g}{2}\big(\mathcal{H}_{ij}\mathcal{H}_{kl} + \mathcal{H}_{ik}\mathcal{H}_{jl} + \mathcal{H}_{jk}\mathcal{H}_{il}\big)\dot{\lambda}, \qquad \lambda = \frac{b}{q}, \tag{3.2}$$

in terms of the lengthened angular metric tensor $\mathcal{H}_{ij}$ (defined by (1.8)).

The representation (3.2) is beautiful but only suggestive, assigning a necessary condition. To gain the full sufficient condition for the $\mathcal{FF}_g^{PD}$–space to be of a Landsberg type, we are to straightforwardly calculate the object $\dot{A}_i$, taking into account the constancy (1.6) of the Finsleroid charge and the vanishing $b^j \nabla_i b_j = 0$ (see (1.13)). On so doing, we arrive at merely

$$\dot{A}_i = \frac{1}{q^2 K}\Big[(b+gq)A_i - \frac{N}{2}gql_i\Big]y^k P_k + \frac{N}{2}\frac{g}{q}P_i \tag{3.3}$$

with P_i being given by (1.11). This representation (3.3) can readily be rewritten in the form (1.10).

It is easy to prove the following.

Lemma. *In the $\mathcal{FF}_g^{PD}$–space with $g = const \neq 0$,*

$$\dot{A}_i = 0 \quad \Longleftrightarrow \quad \nabla_j b_i = k(x)(a_{ij} - b_i b_j). \tag{3.4}$$

Indeed, the right–hand side of the obtained representation (3.3) for the object $\dot{A}_i$ is such that the vanishing $\dot{A}_i = 0$ takes place if and only if the vector P_i is a linear combination of $u_i = a_{ij}y^j$ and b_i:

$$P_i = k_1 u_i + k_2 b_i \tag{3.5}$$

(the tensor $\mathcal{H}_{ik}$ is of the rank $N - 2$ and at each point $x \in M$ has the isotropic plane spanned by the vectors $u_i = a_{ij}y^j$ and b_i because of the identities (1.9)). To agree with the identity $b^i P_i = 0$ (see (1.14)), we must put $k_2 = -bk_1$, obtaining

$$y^j \nabla_j b_i + \frac{1}{2}gqb^j \nabla_j b_i = k_1(u_i - bb_i) \equiv k_1 r_{ij}y^j. \tag{3.6}$$

If we apply here the operator $b^k \partial/\partial y^k$ and take into account the vanishing $r_{ij}b^j = 0$ (see (2.7)), together with $b^k \partial q/\partial y^k = 0$ (which is tantamount to (2.7)), we may conclude that $b^j \nabla_j b_i = 0$, whence in (3.6) the term $y^j \nabla_j b_i$ must be proportional to $u_i - bb_i$. Therefore, the examined condition $\dot{A}_i = 0$ reduces to $y^j \nabla_j b_i = k(u_i - bb_i)$, where k is a factor. Since here the vectors $y^j \nabla_j b_i$ and $u_i - bb_i = (a_{ij} - b_i b_j)y^j$ are linear functions of the set $\{y^j\}$, the factor k may not depend on y^j. Therefore, $y^j \nabla_j b_i = k(x)(u_i - bb_i)$. Differentiating the last result with respect to y^j just yields $\nabla_j b_i = k(a_{ij} - b_i b_j)$ with $k = k(x)$. Lemma is valid.

Therefore, Theorems 3 and 4 are valid.

In processing calculations, it is frequently necessary to keep in mind that the formulas (1.24)–(1.26) entail the following.

Proposition 5. *In any Landsberg case of the $\mathcal{FF}_g^{PD}$–space, the covariant derivative of the input Riemannian metric tensor a_{mn} is such that*

$$a_{mn|k} = -\frac{gk}{q}\Big[(a_{km} - b_k b_m)(u_n - bb_n) + (a_{kn} - b_k b_n)(u_m - bb_m) + (a_{nm} - b_n b_m)(u_k - bb_k)$$

$$-\frac{1}{q^2}(u_m - bb_m)(u_n - bb_n)(u_k - bb_k)\bigg]. \tag{3.7}$$

For the vector $u_m = a_{mn}y^n$ we get

$$u_{m|k} = a_{mn|k}y^n = -\frac{gk}{q}\bigg[q^2(a_{km} - b_k b_m) + (u_k - bb_k)(u_m - bb_m)\bigg] = u_{k|m}, \tag{3.8}$$

after which we obtain

$$b_{|k} = y^n \nabla_k b_n = k(u_k - bb_k), \tag{3.9}$$

$$(q^2 + b^2)_{|k} = a_{mn|k}y^m y^n = -2gq\nabla_n b, \tag{3.10}$$

$$q_{|k} = -\frac{1}{q}(b + gq)\nabla_k b, \tag{3.11}$$

$$(bq)_{|k} = \frac{1}{q}[q^2 - b(b + gq)]\nabla_k b, \qquad (q/b)_{|k} = -\frac{B}{qb^2}\nabla_k b, \qquad (b/q)_{|k} = \frac{B}{q^3}\nabla_k b, \tag{3.12}$$

and

$$J_{|n} = \frac{g}{2q}J\nabla_n b, \qquad B_{|n} = -\frac{gB}{q}\nabla_n b. \tag{3.13}$$

This leads to

$$K\dot{b} = kq^2, \qquad \dot{\lambda} = \frac{B}{q^3}\dot{b}, \tag{3.14}$$

and

$$S\dot{S} = -gq\dot{b}, \quad \dot{q} = -\frac{1}{q}(b + gq)\dot{b}, \quad (\dot{bq}) = \frac{1}{q}[q^2 - b(b + gq)]\dot{b}, \quad \dot{B} = -\frac{gB}{q}\dot{b}. \tag{3.15}$$

It is the implication (3.14) that turns the suggestive representation (3.2) into the conclusive result (1.18).

We have

$$B_{|m|n} - B_{|n|m} = 0 \quad \Longleftrightarrow \quad A_k R^k{}_{mn} = 0. \tag{3.16}$$

4. Study of Curvature Tensor

When straightforwardly calculating the hh-curvature tensor $R^i{}_k$ on the basis of the known definition (see (A.28) in Appendix A below) with $\gamma^i{}_{nm}y^n y^m$ given by (1.24), we obtain the following.

Theorem 5. *In the Landsberg–case $\mathcal{FF}_g^{PD}$–space the following explicit and simple representation is valid:*

$$K^2 R^i{}_k = \frac{g^2 k^2}{4}q^2 \mathcal{H}^i{}_k + gqv^i\widetilde{k}_k - \frac{g}{2q}(\widetilde{k}_n y^n)(v^i v_k + q^2 r^i{}_k) + y^n a_n{}^i{}_{km}y^m. \tag{4.1}$$

Here, $v^i = y^i - bb^i$ and $v_m = u_m - bb_m = r_{mn}y^n$ with $u_m = a_{mn}y^n$; $r_{mn} = a_{mn} - b_m b_n$ being the tensor (2.4); $r^i{}_n = a^{im}r_{mn} = \delta^i{}_n - b^i b_n$; $a_m{}^i{}_{kn}$ is the Riemannian curvature tensor associated with the input Riemannian metric $\mathcal{S}$; $\widetilde{k}_n$ is the vector (1.28). The Berwald case in this theorem is conditioned by $k = 0$.

From (4.1) we find the tensor

$$KR^i{}_{km} = \frac{g^2 k^2}{4}(r^i{}_k v_m - v_k r^i{}_m) + \frac{g}{2q}v^i(\widetilde{k}_k v_m - \widetilde{k}_m v_k) - \frac{1}{2}gq(r^i{}_k \widetilde{k}_m - r^i{}_m \widetilde{k}_k) + y^n a_n{}^i{}_{km} \quad (4.2)$$

(the use of the formula (A.18) of Appendix A is convenient), and then the full hh-curvature tensor

$$R_n{}^i{}_{km} = \frac{g^2 k^2}{4}(r_{nm}r^i{}_k - r_{nk}r^i{}_m) - \frac{g}{2q}v_n\left(\mathcal{H}^i{}_k \widetilde{k}_m - \mathcal{H}^i{}_m \widetilde{k}_k\right)$$

$$+ \frac{g}{2q}\left[r^i{}_n(\widetilde{k}_k v_m - \widetilde{k}_m v_k) + v^i(\widetilde{k}_k r_{mn} - \widetilde{k}_m r_{kn})\right] + a_n{}^i{}_{km} \quad (4.3)$$

(apply the rules (A.29) and (A.30) of Appendix A below). The first term in the right–hand side of (4.3) is independent of vectors y^i. The equalities

$$R^i{}_{km} = l^n R_n{}^i{}_{km} \quad (4.4)$$

and

$$R^i{}_k = l^n R_n{}^i{}_{km}l^m \quad (4.5)$$

hold.

From (4.1) it follows that the *Ricci scalar Ric* $:= R^i{}_i$ is of the value

$$Ric = \frac{1}{K^2}\left[\frac{g^2 k^2}{4}q^2(N-2) - \frac{1}{2}gqN(\widetilde{k}_n y^n) + gq[\widetilde{k}_j y^j - b(b^j \widetilde{k}_j)] + y^n a_n{}^i{}_{im}y^m\right]; \quad (4.6)$$

from (4.3) it follows that the tensor

$$R_{ik} := l^n R_{nikm}l^m \quad (4.7)$$

can explicitly be written as

$$BR_{ik} = \left(\frac{g^2 k^2}{4} - \frac{g}{2q}(\widetilde{k}_m y^m)\right)(q^2 r_{ik} - v_i v_k) + y^n a_{nikm}y^m \quad (4.8)$$

(with $a_{nikm} = a_{ij}a_n{}^j{}_{km}$), which can also be written in terms of the lengthened angular metric tensor as follows:

$$K^2 R_{ik} = \left(\frac{g^2 k^2}{4} - \frac{g}{2q}(\widetilde{k}_m y^m)\right)q^2 \mathcal{H}_{ik} + y^n a_{nikm}y^m \frac{K^2}{B}. \quad (4.9)$$

The symmetry

$$R_{ik} = R_{ki} \quad (4.10)$$

and the identity

$$y^i R_{ik} = 0 \quad (4.11)$$

are obviously valid.

From (4.3) the contracted tensor is found:

$$R_n{}^i{}_{im} = \frac{g^2 k^2}{4}(N-2)r_{nm} - \frac{1}{2}\frac{g}{q}N v_n \widetilde{k}_m$$

$$+\frac{g}{2q}\left[[\widetilde{k}_n - b_n(b^j\widetilde{k}_j)]v_m + [\widetilde{k}_m - b_m(b^j\widetilde{k}_j)]v_n\right] + \frac{g}{2q}\left[(y^j\widetilde{k}_j) - b(b^j\widetilde{k}_j)\right](r_{nm} - \frac{1}{q^2}v_n v_m) + a_n{}^i{}_{im}, \tag{4.12}$$

entailing the following expression for the total contraction of the curvature tensor:

$$R^{nm}{}_{mn} = g^{nm}R_n{}^i{}_{im} = \frac{1}{K^2}\left[\frac{g^2 k^2}{4}(N-2)\Big((N-1)B + g^2 q^2\Big)\right.$$

$$\left.-\frac{1}{2}g^2(N-2)(b+gq)[\widetilde{k}_j y^j - b(b^j\widetilde{k}_j)] + \frac{1}{2}g^2 q^2 N(\widetilde{k}_h b^h)\right] + a_n{}^i{}_{im}g^{nm} \tag{4.13}$$

with

$$a_n{}^i{}_{im}g^{nm} = \frac{B}{K^2}\left(a_n{}^i{}_{im}a^{nm} - \frac{gb}{q}(N-1)(\widetilde{k}_h b^h) + \frac{2g}{q}[(N-1)(\widetilde{k}_h y^h) - (\widetilde{k}_h v^h)]\right)$$

$$+\frac{1}{K^2}\frac{g}{q}(b+gq)y^n a_n{}^i{}_{im}y^m. \tag{4.14}$$

We have

$$R_n{}^i{}_{im}y^n y^m = K^2 Ric. \tag{4.15}$$

The tensor (4.12) is non–symmetric:

$$R_n{}^i{}_{im} - R_m{}^i{}_{in} = -\frac{1}{2}\frac{g}{q}N(v_n\widetilde{k}_m - v_m\widetilde{k}_n). \tag{4.16}$$

The contraction

$$R_n{}^i{}_{im}y^m = \frac{g^2 k^2}{4}(N-2)v_n - \frac{1}{2}\frac{g}{q}N v_n(\widetilde{k}_m y^m)$$

$$+\frac{g}{2q}\left[[\widetilde{k}_n - b_n(b^j\widetilde{k}_j)]q^2 + [\widetilde{k}_m y^m - b(b^j\widetilde{k}_j)]v_n\right] + a_n{}^i{}_{im}y^m \tag{4.17}$$

is obtained.

Let us consider also the *Ricci tensor*

$$Ric_{nm} := \frac{1}{2}\frac{\partial^2(K^2 Ric)}{\partial y^n \partial y^m}, \tag{4.18}$$

the *Ricci–deflection vector*

$$\Upsilon_n := \frac{1}{2}\frac{\partial(K^2 Ric)}{\partial y^n} - R_n{}^i{}_{im}y^m \tag{4.19}$$

and the *Ricci–deflection tensor*

$$\Upsilon_{nm} := Ric_{nm} - \frac{1}{2}(R_n{}^i{}_{im} + R_m{}^i{}_{in}) \equiv \Upsilon_{mn}. \tag{4.20}$$

We obtain upon direct calculations the following lucid representations in terms of the lengthened angular metric tensor:

$$\Upsilon_n = -\frac{1}{4}gqN\mathcal{H}_n{}^m\widetilde{k}_m \tag{4.21}$$

and

$$\Upsilon_{nm} = -\frac{1}{4}\frac{g}{q}N(\widetilde{k}_j y^j)\mathcal{H}_{nm}. \tag{4.22}$$

The identities

$$\Upsilon_n y^n = 0, \qquad \Upsilon_n A^n = 0, \qquad \Upsilon_n b^n = 0 \tag{4.23}$$

and

$$\Upsilon_{nm} y^m = 0, \qquad \Upsilon_{nm} A^m = 0, \qquad \Upsilon_{nm} b^m = 0 \tag{4.24}$$

are valid. It is useful to compare (4.21) and (4.22) with (1.10) and (1.19).

To lower the index in the curvature tensor (4.3) we must apply the explicit expression (see (A.2) in Appendix A below) for the Finsleroid metric tensor. On so doing, and using the contractions

$$b_j R_n{}^j{}_{km} = b_j a_n{}^j{}_{km} = -\widetilde{k}_k r_{nm} + \widetilde{k}_m r_{nk}$$

(see (1.27)) and

$$u_j R_n{}^j{}_{km} = \frac{g^2 k^2}{4}(r_{nm}v_k - r_{nk}v_m) + \frac{1}{2}gq[\widetilde{k}_k(r_{nm} + \frac{1}{q^2}v_n v_m) - \widetilde{k}_m(r_{nk} + \frac{1}{q^2}v_n v_k)] + u_j a_n{}^j{}_{km},$$

we obtain the result

$$\frac{B}{K^2}R_{nikm} = \frac{g^2 k^2}{4}(r_{nm}r_{ik} - r_{nk}r_{im}) - \frac{1}{2}\frac{g}{q}v_n\left((r_{ik} - \frac{1}{q^2}v_i v_k)\widetilde{k}_m - (r_{im} - \frac{1}{q^2}v_i v_m)\widetilde{k}_k\right)$$

$$+ \frac{g}{2q}\left[r_{in}(\widetilde{k}_k v_m - \widetilde{k}_m v_k) + v_i(\widetilde{k}_k r_{mn} - \widetilde{k}_m r_{kn})\right] + a_{nikm}$$

$$+ \frac{g}{qB}\left[(gq^3 b_i + S^2 v_i)(-\widetilde{k}_k r_{nm} + \widetilde{k}_m r_{nk})\right.$$

$$+ (S^2 b_i - bu_i)\left(\frac{g^2 k^2}{4}(r_{nm}v_k - r_{nk}v_m) + \frac{1}{2}gq[\widetilde{k}_k(r_{nm} + \frac{1}{q^2}v_n v_m) - \widetilde{k}_m(r_{nk} + \frac{1}{q^2}v_n v_k)] + u_j a_n{}^j{}_{km}\right)\bigg],$$

which can be simplified to read

$$\frac{B}{K^2}R_{nikm} = \frac{g^2 k^2}{4}\left[(r_{nm}r_{ik} - r_{nk}r_{im}) + \frac{g}{Bq}(S^2 b_i - bu_i)(r_{nm}v_k - r_{nk}v_m)\right]$$

$$+ \frac{g}{2q}\left[(r_{in} - \frac{1}{q^2}v_i v_n)(\widetilde{k}_k v_m - \widetilde{k}_m v_k) - v_i(\widetilde{k}_k r_{mn} - \widetilde{k}_m r_{kn})\right] - \frac{1}{2}\frac{g}{q}v_n(r_{ik}\widetilde{k}_m - r_{im}\widetilde{k}_k)$$

$$- \frac{g^2}{2B}(S^2 b_i - bu_i)\left[\widetilde{k}_k(r_{nm} - \frac{1}{q^2}v_n v_m) - \widetilde{k}_m(r_{nk} - \frac{1}{q^2}v_n v_k)\right]$$

$$+\frac{g}{qB}(S^2 b_i - bu_i)y^j a_{njkm} + a_{nikm}. \tag{4.25}$$

Let us use this representation to evaluate the contracted tensor $g^{nm}R_{nikm}$ (the contravariant components of the metric tensor involved are given by (A.3) in Appendix A below). We get in succession:

$$\frac{B}{K^2}a^{nm}R_{nikm} = \frac{g^2 k^2}{4}(N-2)\left[r_{ik} + \frac{g}{Bq}(S^2 b_i - bu_i)v_k\right]$$

$$+\frac{g}{2q}\left[-(r_i{}^m - \frac{1}{q^2}v_i v^m)\widetilde{k}_m v_k - v_i(\widetilde{k}_k(N-2) - \widetilde{k}_m r^m{}_k)\right] - \frac{1}{2}\frac{g}{q}r_{ik}(\widetilde{k}_m v^m)$$

$$-\frac{g^2}{2B}(S^2 b_i - bu_i)\left[\widetilde{k}_k(N-2) - (r_k{}^m - \frac{1}{q^2}v_k v^m)\widetilde{k}_m\right]$$

$$+a^{nm}a_{nikm} + \frac{g}{qB}(S^2 b_i - bu_i)y^j a_{njkm}a^{nm}$$

and

$$\frac{B}{K^2}y^m y^n R_{nikm} = \frac{g^2 k^2}{4}(q^2 r_{ik} - v_i v_k) - \frac{1}{2}gq(r_{ik} - \frac{1}{q^2}v_i v_k)(\widetilde{k}_m y^m) + y^m y^n a_{nikm}$$

(which confirms (4.8)), together with

$$\frac{B}{K^2}y^m b^n R_{nikm} = y^m b^n a_{nikm} + \frac{g}{qB}(S^2 b_i - bu_i)u_j a_n{}^j{}_{km}y^m b^n$$

$$= v_i \widetilde{k}_k - (\widetilde{k}_j y^j)r_{ik} + \frac{g}{qB}(S^2 b_i - bu_i)[q^2 \widetilde{k}_k - (\widetilde{k}_j y^j)v_k]$$

and

$$\frac{B}{K^2}b^m y^n R_{nikm} = -\frac{1}{2}gq(r_{ik} - \frac{1}{q^2}v_i v_k)(\widetilde{k}_m b^m) + v_k \widetilde{k}_i - (\widetilde{k}_j y^j)r_{ik}$$

supplemented by

$$\frac{B}{K^2}b^m b^n R_{nikm} = b^m b^n a_{nikm} + \frac{g}{qB}(S^2 b_i - bu_i)u_j a_n{}^j{}_{km}b^m b^n$$

$$= -(\widetilde{k}_j b^j)r_{ik} - \frac{g}{qB}(S^2 b_i - bu_i)v_k(\widetilde{k}_j b^j).$$

The result of calculations is the following:

$$g^{nm}R_{nikm} = \frac{g^2 k^2}{4}\left[(N-2)\left[r_{ik} + \frac{g}{Bq}(S^2 b_i - bu_i)v_k\right] + \frac{g}{qB}(b+gq)(q^2 r_{ik} - v_i v_k)\right]$$

$$+\frac{g}{2q}\left[-(r_i{}^m - \frac{1}{q^2}v_i v^m)\widetilde{k}_m v_k - v_i(\widetilde{k}_k(N-2) - \widetilde{k}_m r^m{}_k)\right] - \frac{1}{2}\frac{g}{q}r_{ik}(\widetilde{k}_m v^m)$$

$$-\frac{g^2}{2B}(S^2 b_i - b u_i)\Big[\widetilde{k}_k(N-2) - (r_k{}^m - \frac{1}{q^2}v_k v^m)\widetilde{k}_m\Big]$$

$$-\frac{g^2}{2B}(b + gq)(r_{ik} - \frac{1}{q^2}v_i v_k)(\widetilde{k}_m y^m)$$

$$-\frac{g}{q}\left[v_i \widetilde{k}_k - (\widetilde{k}_j y^j)r_{ik} + \frac{g}{qB}(S^2 b_i - b u_i)[q^2 \widetilde{k}_k - (\widetilde{k}_j y^j)v_k]\right]$$

$$-\frac{g}{q}\left[-\frac{1}{2}gq(r_{ik} - \frac{1}{q^2}v_i v_k)(\widetilde{k}_m b^m) + v_k \widetilde{k}_i - (\widetilde{k}_j y^j)r_{ik}\right]$$

$$+\frac{gb}{q}\left[-(\widetilde{k}_j b^j)r_{ik} - \frac{g}{qB}(S^2 b_i - b u_i)v_k(\widetilde{k}_j b^j)\right]$$

$$+a^{nm}a_{nikm} + \frac{g}{qB}\Big[(S^2 b_i - b u_i)y^j a_{njkm}a^{nm} + (b + gq)y^m y^n a_{nikm}\Big].$$

Simplifying yields

$$g^{nm}R_{nikm} = \frac{g^2 k^2}{4}\left[(N-2)\Big[r_{ik} + \frac{g}{Bq}(S^2 b_i - b u_i)v_k\Big] + \frac{g}{qB}(b + gq)(q^2 r_{ik} - v_i v_k)\right]$$

$$+\frac{g}{2q}\left[-(r_i{}^m - \frac{1}{q^2}v_i v^m)\widetilde{k}_m v_k - v_i(\widetilde{k}_k N - \widetilde{k}_m r^m{}_k)\right] + \frac{1}{2}\frac{g}{q}r_{ik}(\widetilde{k}_m v^m)$$

$$-\frac{g^2}{2B}(S^2 b_i - b u_i)\Big[\widetilde{k}_k N - (r_k{}^m + \frac{1}{q^2}v_k v^m)\widetilde{k}_m\Big]$$

$$-\frac{g^2}{2B}(b + gq)(r_{ik} - \frac{1}{q^2}v_i v_k)(\widetilde{k}_m y^m)$$

$$+\frac{g}{q}(\widetilde{k}_j y^j)r_{ik} + \frac{1}{2}g^2(r_{ik} - \frac{1}{q^2}v_i v_k)(\widetilde{k}_m b^m) - \frac{g}{q}v_k \widetilde{k}_i$$

$$+a^{nm}a_{nikm} + \frac{g}{qB}\Big[(S^2 b_i - b u_i)y^j a_{njkm}a^{nm} + (b + gq)y^m y^n a_{nikm}\Big]. \qquad (4.26)$$

Whenever $\widetilde{k}_n = f b_n$, we obtain

$$g^{nm}R_{nikm} = \frac{g^2 k^2}{4}\left[(N-2)\Big[r_{ik} + \frac{g}{Bq}(S^2 b_i - b u_i)v_k\Big] + \frac{g}{qB}(b + gq)(q^2 r_{ik} - v_i v_k)\right]$$

$$-\frac{gN}{2q}v_i b_k f - \frac{g^2}{2B}(S^2 b_i - bu_i)b_k N f - \frac{g^2}{2B}b(b + gq)(r_{ik} - \frac{1}{q^2}v_i v_k)f$$

$$+\frac{gb}{q}r_{ik}f + \frac{1}{2}g^2(r_{ik} - \frac{1}{q^2}v_i v_k)f - \frac{g}{q}v_k b_i f$$

$$+a^{nm}a_{nikm} + \frac{g}{qB}\left[(S^2 b_i - bu_i)y^j a_{njkm}a^{nm} + (b + gq)y^m y^n a_{nikm}\right], \qquad (4.27)$$

$$R_i{}^h{}_{hk} = \frac{g^2 k^2}{4}(N - 2)r_{ik} - \frac{gN}{2q}v_i b_k f + a_i{}^h{}_{hk}, \qquad (4.28)$$

and

$$R^{nm}{}_{mn} = \frac{1}{K^2}\left[\frac{g^2 k^2}{4}(N - 2)\Big((N - 1)B + g^2 q^2\Big) + \frac{1}{2}g^2 q^2 N f\right] + a_n{}^h{}_{hm}g^{nm} \qquad (4.29)$$

with

$$a_n{}^h{}_{hm}g^{nm} = \frac{B}{K^2}\Big(a_n{}^h{}_{hm}a^{nm} + \frac{gb}{q}(N - 1)f\Big) + \frac{1}{K^2}\frac{g}{q}(b + gq)y^n a_n{}^h{}_{hm}y^m. \qquad (4.30)$$

5. Warped Product Structure of Associated Riemannian Space

Remarkably, the Landsberg–characteristic condition (1.15)–(1.16), as well as the Berwald-characteristic condition $\nabla_i b_j = 0$, does not involve the Finsleroid charge parameter g and, therefore, imposes restrictions on only the underlying associated Riemannian space $\mathcal{R}_N = (M, \mathcal{S})$. What is the geometrical meaning of the restrictions?

The fact that the Finsleroid–axis 1-form b is closed (see (1.15)) and the b-lines are geodesics (see (1.23) and Proposition 1) can be used to introduce on the background Riemannian space $\mathcal{R}_N = (M, \mathcal{S})$ the *b-geodesic coordinates*, to be denoted as $z^i = \{z^0, z^a\}$, such that with respect to such coordinates we shall have

$$b^0(z^i) = 1, \qquad b^a(z^i) = 0, \qquad b_0(z^i) = 1, \qquad b_a(z^i) = 0, \qquad (5.1)$$

$$a_{00}(z^i) = 1, \qquad a_{0a}(z^i) = 0, \qquad a_{ab}(z^i) = r_{ab}(z^i), \qquad r_{00}(z^i) = r_{0a}(z^i) = 0 \qquad (5.2)$$

(we have used the unit length (2.5) and the nullification (2.7)), obtaining the square of the Riemannian line element ds of $\mathcal{R}_N = (M, \mathcal{S})$ to be the sum

$$(ds)^2 = (dz^0)^2 + r_{ab}(z^i)dz^a dz^b \qquad (5.3)$$

and the Landsberg–characteristic condition (1.16) to read

$$\frac{1}{2}\frac{\partial r_{ab}(z^i)}{\partial z^0} = k(z^i)r_{ab}(z^i); \qquad (5.4)$$

the indices a, b, c are specified over the range $1, \ldots, N - 1$.

Solving the last equation by the help of the representation

$$r_{ab}(z^i) = \phi^2(z^i)p_{ab}(z^c) \tag{5.5}$$

subject to

$$\frac{\partial \phi}{\partial z^0} = k, \tag{5.6}$$

we can convert the sum (5.3) into the sum

$$(ds)^2 = (dz^0)^2 + \phi^2(z^i)p_{ab}(z^c)dz^a dz^b \tag{5.7}$$

which says us that ds is *a warped product metric*. To emphasize the fact that the product property (5.7) has appeared upon specifying the z^0-coordinate line to be tangent to the Finsleroid–axis vector field $b^i(x)$, we shall attribute to the space and metric obtained the quality of being of the *b-warped product type*. Thus we have the following.

Theorem 6. *An $\mathcal{FF}_g^{PD}$–space is of the Landsberg type if and only if the associated Riemannian space is a b-warped product space and the Finsleroid charge g is a constant.*

The warped product type of the Riemannian space was nicely exposed in the section 13.3 of the book [7] (and we follow the terminology used therein). In that source, the interesting special class of warped products that is characterized by the condition that the function $\phi(z^i)$ be independent of z^c was introduced and considered. Accordingly, we call the associated Riemannian space *special b-warped space* if the sum (5.7) is simplified to be

$$(ds)^2 = (dt)^2 + \phi^2(t)p_{ab}(z^c)dz^a dz^b, \tag{5.8}$$

where $t = z^0$. In this case from (5.4)–(5.6) it follows that the function k, and hence the field $\widetilde{k}_n = \dfrac{\partial k}{\partial x^n} + k^2 b_n$ introduced in accordance with (1.28), may depend on only $t = z^0$. Therefore, when treating with respect to the b-geodesic coordinates, we have the proportionality of $\widetilde{k}_n$ to b_n. However, the last proportionality is tensorial and must remain valid in terms of any admissible coordinates. Thus the following theorem may be stated to hold.

Theorem 7. *An $\mathcal{FF}_g^{PD}$–space of the Landsberg type is of the special b-warped product type if and only if the 1-form $\widetilde{k}_n(x)y^n$ is proportional to the 1-form $b = b_n(x)y^n$:*

$$\widetilde{k}_n(x) = f(x)b_n(x). \tag{5.9}$$

It is the condition (5.9) under which we have obtained the simplified representations (4.27)–(4.30) in the previous section.

It is also interesting to specialize the Landsberg–type $\mathcal{FF}_g^{PD}$–space farther by the help of the following definition: The Landsbergian $\mathcal{FF}_g^{PD}$–space is called *b-stationary* if

$$b_i a_n{}^i{}_{km} = 0 \tag{5.10}$$

holds at any point $x \in M$. In this case, given a $\mathcal{FF}_g^{PD}$–space of the b-stationary Landsberg type provided the space is not of the Berwald type, the identities

$$A_i R^i{}_k = 0, \qquad A_i R^i{}_{km} = 0 \tag{5.11}$$

hold, k is not a constant, the equalities

$$b_n = \frac{\partial}{\partial x^n} \frac{1}{k}, \qquad \widetilde{k}_n = 0 \tag{5.12}$$

are fulfilled, and the curvature tensor (4.3) is entirely independent of tangent vectors y. With respect to the b-stationary coordinates, from (5.1) and (5.12) we obtain $k = 1/t$. If we specify the Landsberg type of the $\mathcal{FF}_g^{PD}$–space by stipulating that

$$(\nabla_m \nabla_n - \nabla_n \nabla_m)b_k = 0, \tag{5.13}$$

then, noting (1.26), we may re-interpret this commutator in terms of the Finsler–nature covariant derivatives to read

$$b_{k|n|m} - b_{k|m|n} = 0, \tag{5.14}$$

so that the identity (5.10) together with

$$b_i R_n{}^i{}_{km} = 0 \tag{5.15}$$

and

$$b_i R^i{}_{km} = 0, \qquad b_i R^i{}_k = 0 \tag{5.16}$$

(see (1.27)) must appear. Since the vanishing $y_i R^i{}_{km} = y_i R^i{}_k = 0$ takes place in any Finsler space, and A_i are linear combinations of y_i and b_i (see (A.7) in Appendix A), the contraction of the tensors $R^i{}_k$ and $R^i{}_{km}$ by A_i must produce zeros, whence we arrive at (5.11). Finally, in view of (1.27), the stipulation (5.13) entails the vanishing (5.12). The opposite way, namely derivation of the formulas (5.11)–(5.16) from the stationarity condition (5.10), can easily be gone, too. Each of the above formulas (5.11)–(5.16) can be taken as the condition that may characterize the b-stationary Landsberg case instead of the departure condition (5.10).

6. Indefinite (Relativistic) Case

The positive–definite case (to which the previous four sections were devoted) can be juxtaposed with a indefinite pseudo–Finsleroid case, assuming the input metric tensor $\{a_{ij}(x)\}$ to be pseudo–Riemannian with the *time–space signature:*

$$\mathrm{sign}(a_{ij}) = (+ - - \ldots). \tag{6.1}$$

Namely, attempting to generalize the pseudo–Riemannian geometry in a pseudo–Finsleroid Finslerian way, we are to adapt the consideration to the following decomposition of the tangent bundle TM:

$$TM = \mathcal{S}_g^+ \cup \Sigma_g^+ \cup \mathcal{R}_g \cup \Sigma_g^- \cup \mathcal{S}_g^-, \tag{6.2}$$

which sectors relate to the cases that the tangent vectors $y \in TM$ are, respectively, time–like, upper–cone isotropic, space–like, lower–cone isotropic, or past–like. The sectors are defined according to the following list:

$$\mathcal{S}_g^+ = \Big(y \in \mathcal{S}_g^+ : \ y \in T_x M, \ b(x,y) > -g_-(x)q(x,y) \Big), \tag{6.3}$$

$$\Sigma_g^+ = \left(y \in \Sigma_g^+ : \ y \in T_x M, \ b(x,y) = -g_-(x)q(x,y) \right), \tag{6.4}$$

$$\mathcal{R}_g^+ = \left(y \in \mathcal{R}_g^+ : \ y \in T_x M, \ -g_-(x)q(x,y) > b(x,y) > 0 \right), \tag{6.5}$$

$$\mathcal{R}^0 = \left(y \in \mathcal{R}^0 : \ y \in T_x M, \ b(x,y) = 0 \right), \tag{6.6}$$

$$\mathcal{R}_g^- = \left(y \in \mathcal{R}_g^- : \ y \in T_x M, \ 0 > b(x,y) > -g_+(x)q(x,y) \right), \tag{6.7}$$

$$\Sigma_g^- = \left(y \in \Sigma_g^- : \ y \in T_x M, \ b(x,y) = -g_+(x)q(x,y) \right), \tag{6.8}$$

$$\mathcal{S}_g^- = \left(y \in \mathcal{S}_g^- : \ y \in T_x M, \ b(x,y) < -g_+(x)q(x,y) \right), \tag{6.9}$$

$$\mathcal{R}_g = \mathcal{R}_g^+ \cup \mathcal{R}_g^- \cup \mathcal{R}^0. \tag{6.10}$$

We use the convenient notation

$$G = \frac{g}{h}, \qquad h = \sqrt{1 + \frac{1}{4}g^2} \tag{6.11}$$

(instead of (2.10)),

$$g_+ = -\frac{1}{2}g + h, \qquad g_- = -\frac{1}{2}g - h, \tag{6.12}$$

$$G_+ = \frac{g_+}{h} \equiv -\frac{1}{2}G + 1, \qquad G_- = \frac{g_-}{h} \equiv -\frac{1}{2}G - 1, \tag{6.13}$$

$$g^+ = \frac{1}{g_+} = -g_-, \qquad g^- = \frac{1}{g_-} = -g_+, \tag{6.14}$$

$$g^+ = \frac{1}{2}g + h, \qquad g^- = \frac{1}{2}g - h, \tag{6.15}$$

$$G^+ = \frac{g^+}{h} \equiv \frac{1}{2}G + 1, \qquad G^- = \frac{g^-}{h} \equiv \frac{1}{2}G - 1. \tag{6.16}$$

The following identities hold

$$g_+ + g_- = -g, \qquad g_+ - g_- = 2h, \tag{6.17}$$

$$g^+ + g^- = g, \qquad g^+ - g^- = 2h, \tag{6.18}$$

$$g_+ g_- = -1, \qquad g^+ g^- = -1, \tag{6.19}$$

together with the *g–symmetry*

$$g_+ \overset{g \to -g}{\Longleftrightarrow} -g_-, \qquad g^+ \overset{g \to -g}{\Longleftrightarrow} -g^-, \qquad G_+ \overset{g \to -g}{\Longleftrightarrow} -G_-, \qquad G^+ \overset{g \to -g}{\Longleftrightarrow} -G^-. \tag{6.20}$$

It is implied that $g = g(x)$ is a scalar on the underlying manifold M. All the range

$$-\infty < g(x) < \infty \tag{6.21}$$

(instead of (2.9)) is now admissible. We also assume that the manifold M admits a normalized 1-form $b = b(x, y)$ which is *timelike* in terms of the pseudo–Riemannian metric $\mathcal{S}$, such that the pseudo–Riemannian length of the involved vector b_i be equal to 1. With respect to natural local coordinates in the space $\mathcal{R}_N$ we have the local representations

$$b = b_i(x)y^i, \tag{6.22}$$

$$S = \sqrt{|a_{ij}(x)y^i y^j|}, \tag{6.23}$$

$$q = \sqrt{|r_{ij}(x)y^i y^j|}, \tag{6.24}$$

$$r_{ij}(x) = b_i(x)b_j(x) - a_{ij}(x), \tag{6.25}$$

$$a^{ij}b_i b_j = 1, \tag{6.26}$$

$$S^2 = b^2 - q^2, \tag{6.27}$$

$$b^i r_{ij} = 0, \tag{6.28}$$

where

$$b^i := a^{ik}b_k \tag{6.29}$$

(compare with (2.1)–(2.8)).

The *pseudo–Finsleroid characteristic quadratic form*

$$B(x, y) := b^2 - gqb - q^2 \equiv (b + g_+q)(b + g_-q) \tag{6.30}$$

is now of the positive discriminant

$$D_{\{B\}} = 4h^2 > 0 \tag{6.31}$$

(compare with (2.11) and (2.12)).

In terms of these concepts, we propose

Definition. The scalar function $F(x, y)$ given by the formulas

$$F(x, y) := \sqrt{|B(x, y)|}\, J(x, y) \equiv |b + g_-q|^{G_+/2}|b + g_+q|^{-G_-/2} \tag{6.32}$$

and

$$J(x, y) = \left|\frac{b + g_-q}{b + g_+q}\right|^{-G/4}, \tag{6.33}$$

is called the *pseudo–Finsleroid–Finsler metric function.*

The positive (not absolute) homogeneity holds: $F(x, \lambda y) = \lambda F(x, y)$ for any $\lambda > 0$. The functions

$$L(x, y) = q - \frac{g}{2}b \qquad (6.34)$$

and

$$A(x, y) = b - \frac{g}{2}q \qquad (6.35)$$

are now to be used instead of (2.18) and (2.19), so that (2.20) changes to read

$$L^2 - h^2 b^2 = B, \qquad A^2 - h^2 q^2 = B. \qquad (6.36)$$

Similarly to (2.21), we introduce

Definition. The arisen space

$$\mathcal{FF}_g^{SR} := \{\mathcal{R}_N;\ b(x, y);\ g(x);\ F(x, y)\} \qquad (6.37)$$

is called the *pseudo–Finsleroid–Finsler space*.

The upperscript "SR" emphasizes the Specially Relativistic character of the space under study.

Definition. The space $\mathcal{R}_N = (M, \mathcal{S})$ entering the above definition is called the *associated pseudo–Riemannian space*.

Definition. The scalar $g(x)$ is called the *pseudo–Finsleroid charge*. The 1-form b is called the *pseudo–Finsleroid–axis* 1-*form*.

It can be verified that the Finslerian metric tensor constructed from the function F given by (6.32) does inherit from the tensor $\{a_{ij}(x)\}$ the time–space signature (6.1):

$$\mathrm{sign}(g_{ij}) = (+ - - \dots). \qquad (6.38)$$

The structure (2.27) for the Cartan tensor remains valid in the pseudo–Finsleroid case, now with

$$A_h A^h = -\frac{N^2}{4}g^2. \qquad (6.39)$$

Elucidating the structure of the respective indicatrix curvature tensor (2.29) of the $\mathcal{FF}_g^{SR}-$ space again results in the special type (2.30), with $S^* = \frac{1}{4}g^2$, so that

$$\mathcal{R}_{\text{pseudo-Finsleroid Indicatrix}} = -(1 + \frac{1}{4}g^2) \leq -1$$

(compare with (2.32)) and

$$\mathcal{R}_{\text{pseudo-Finsleroid Indicatrix}} \xEqualunder{g \to 0} \mathcal{R}_{\text{pseudo-Euclidean Sphere}} = -1.$$

The pseudo–Finsleroid indicatrix is a space of constant negative curvature.
By analogy with (2.33) and (2.34) we obtain the $\mathcal{FF}_g^{SR}-$*scalar product*

$$< y_1, y_2 >_{\{x\}} \ := F(x, y_1)F(x, y_2)\cosh\Big(\alpha_{\{x\}}(y_1, y_2)\Big), \qquad y_1, y_2 \in \mathcal{S}_g^+, \qquad (6.40)$$

and the $\mathcal{F}\mathcal{F}_g^{SR}-angle$

$$\alpha_{\{x\}}(y_1, y_2) \; := \; \frac{1}{h} \operatorname{arccosh} \frac{A(x, y_1)A(x, y_2) - h^2 < y_1, y_2 >_{\{x\}}^{\{r\}}}{\sqrt{|B(x, y_1)|} \sqrt{|B(x, y_2)|}}, \qquad y_1, y_2 \in \mathcal{S}_g^+, \quad (6.41)$$

where again $< y_1, y_2 >_{\{x\}}^{\{r\}} = r_{ij}(x)y_1^i y_2^j$. The $\mathcal{F}\mathcal{F}_g^{SR}-distance$ Δs between ends of the vectors in the tangent space is now given by the formula

$$(\Delta s)^2 = (F(x, y_1))^2 + (F(x, y_2))^2 - 2F(x, y_1)F(x, y_2) \cosh\Big(\alpha_{\{x\}}(y_1, y_2)\Big). \qquad (6.42)$$

Also,

$$< y, y >_{\{x\}} = (F(x, y))^2. \qquad (6.43)$$

The pseudo–Finsleroid analogue of the angle (2.39) between the Finsleroid axis and a vector now reads

$$\alpha_{\{x\}}(y) \; := \; \frac{1}{h} \operatorname{arccosh} \frac{A(x, y)}{\sqrt{|B(x, y)|}}. \qquad (6.44)$$

All the motivation presented in previous Sections 4 and 5 when elucidating the Berwald and Landsberg cases, can be repeated word–for–word in the present indefinite type, with slight changes in formulas.

7. Conclusions

Thus, we have geometrized the tangent bundle by the help of the Finsleroid, *resp.* pseudo–Finsleroid, in the positive–definite approach, *resp.* indefinite approach of the relativistic signature. Under the particular condition $g = const$, the Finsleroid metric function K, *resp.* the pseudo–Finsleroid metric function F, is defined by a 1-form b and a Riemannian, *resp.* pseudo–Riemannian, metric. In general, they involve also a scalar, the "geometric charge" $g(x)$.

Both the metric functions, K defined by (2.14) and F defined by (6.32), are positively, and *not* absolutely, homogeneous of the degree 1: $K(x, \lambda y) = \lambda K(x, y)$, $\lambda > 0$, together with $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$. Also, they are reversible in the extended sense

$$K\Big|_{g \to -g, \ y \to -y} = K, \qquad F\Big|_{g \to -g, \ y \to -y} = F.$$

The traditional symmetry assumption $K(x, -y) = K(x, y)$, as well as $F(x, -y) = F(x, y)$, remains valid for the vectors $y \in T_x M$ obeying $b(x, y) = 0$, and is violating otherwise unless $g = 0$.

Each metric function, K and F, is rotund around the Finsleroid axis and, therefore, reflects the idea of the spherical symmetry in the spaces of directions orthogonal to the axis. Their indicatrices are spaces of constant curvature, namely positive in case of K and negative in case of F.

The Finslerian and pseudo–Finsleroid spaces arisen are naturally endowed with the inner scalar product and the concept of angle.

We have deduced the affirmative and detailed answers to all the two questions posed in the beginning of the present paper. Moreover, we have supplemented the answers by explicit and rather simple representations for the key geometric objects associated

thereto. A particular peculiarity is that the respective Landsberg–case tensor $A_{i|j}$ proves to be proportional to the lengthened angular metric tensor $\mathcal{H}_{ij}$, according to (1.19).

Surprisingly, the tensor $\mathcal{H}_i{}^j$ proves to be entirely independent of the Finsleroid charge g (see (A.16) in Appendix A below), that is of whether the Finsleroid extension of the underlying Riemannian geometry is performed or not. In this respect, the tensor is *universal* and is free tractable in, and coming from, the entire context of the underlying Riemannian space.

Having an $\mathcal{FF}_g^{PD}$–space with constant Finsleroid charge, calculations of the hv–curvature tensor $P_k{}^i{}_{mn}$ yield the result explicitly in terms of the lengthened angular metric tensor $\mathcal{H}_{mn}$ (see (1.18) and the end of Appendix A below) and, in the Landsberg–case thereto, the structure of the associated hh-curvature tensor $R_n{}^i{}_{km}$ can clearly be explained by the help of the explicit representations (4.1)–(4.30) obtained, in particular the Ricci–deflection vector and tensor are remarkably proportional to the tensor $\mathcal{H}_n{}^m$ in accordance with (4.21) and (4.22).

The comprehension of the Landsberg case $\mathcal{FF}_g^{PD}$–space in terms of the category of a warped product structure (to which Section 5 was devoted) seems to be a constructive and handy idea.

The C^∞-class smoothness, though appears frequently in publications devoted to Finsler geometry and might outwardly sound as being a desirable attractive tune, is ordinarily ill-incorporating in designs of particular Finslerian metric functions. Our examples K and F used are not total exclusions from this rule, namely the functions are analytic along any direction except for that which points in, or opposite to, the direction of the Finsleroid axis, that is, when $q \to 0$. This actually does not destroy the potential applied abilities of the Finsleroid metric functions, as long as we exercise a sufficient care when encountering with the Finsleroid–axis–directions. The metric functions K and F and all the components of the associated covariant vector y_i, as well as the determinant (see (A.6) in Appendix A below) of the Finslerian metric tensor, are all everywhere smooth, the divergences (at $q = 0$) start arising with components of the Finslerian metric tensor. In particular, g_{ij} and g^{ij} (see (A.2) and (A.3) in Appendix A below), as well as A_i and A^i (see (A.7) and (A.8) in Appendix A below), are ($\sim 1/q$)-singular, and simultaneously the contraction $A_h A^h$ (see (2.28)) is entirely independent of y. The hv-curvature tensor in the Landsberg case is also singular when $q \to 0$ (see (1.18)), however the very condition (1.16) that underlines the Landsberg case is free of any singularities. On the other hand, the components of the tensor $q^2 \mathcal{H}_i{}^j$ are quadratic in the variables y and, therefore, are analytical in any direction (examine the right–hand side of the representation (A.16) placed in Appendix A below).

By following the book [7], we say that the Finsler space is *y-global* if the space is smooth and strongly convex on all of the *slit tangent bundle* $TM \setminus 0$. In many cases it is important to supplement the notion by explicit indicating the degree of smoothness. The following assertions are valid. The $\mathcal{FF}_g^{PD}$–space is y-global of the class C^1, and not of the class C^2, on all of the $TM \setminus 0$. The $\mathcal{FF}_g^{PD}$–space is y-global of the class C^∞ on all of the *b-slit tangent bundle*

$$\mathcal{T}_b M := TM \setminus 0 \setminus b \setminus -b \qquad (7.1)$$

(obtained by deleting out in $TM \setminus 0$ all the directions which point along, or oppose, the directions given rise to by the Finsleroid–axis 1-form b; the value $q = 0$ just corresponds to such directions in view of the initial formulas (2.3) and (2.7)), provided the C^∞–smoothness is assumed for the input Riemannian space $\mathcal{R}_N = (M, S)$, the 1-form b, and the scalar $g(x)$.

Above our consideration was everywhere referred to local vicinities of the points $x \in M$ of the underlying manifold. Let us ponder over the possibility to construct global $\mathcal{FF}_g^{PD}$–structures. In the Berwald case, that is when $k = 0$, the tensor $r_{ab}(z^i)$ looses the dependence on z^0 (as this is evident from (5.4)), so that the underlying Riemannian space $\mathcal{R}_N = (M, \mathcal{S})$ must be locally the Cartesian product of an $(N - 1)$-dimensional Riemannian space $\mathcal{R}_{N-1}$ and an interval of a one–dimensional Euclidean space $\mathbb{R}$, with the $r_{ab}(z^c)$ playing the role of the Riemannian metric tensor of the former space and the Finsleroid–axis 1–form b being tangent to the latter space. In particular, on the product Riemannian spaces

$$\mathcal{R}_N = \mathcal{R}_{N-1} \times \mathbb{R} \quad \text{and} \quad \mathcal{R}_N = \mathcal{R}_{N-1} \times \mathbb{S}^1, \tag{7.2}$$

the Berwald–case $\mathcal{FF}_g^{PD}$–space structure can be introduced straightforwardly by imposing on such spaces the Finsleroid metric function K in accordance with its very definition (1.14) and assuming the Finsleroid charge g to be a constant. If we modify the construction in but multiplying the metric tensor $r_{ab}(z^c)$ of the involved space $\mathcal{R}_{N-1}$ by a factor $\phi(z^i)$, we just lift the structure from the Berwaldian level to the Landsbergian type, with the function k obtained from (5.6). The Landsberg–type $\mathcal{FF}_g^{PD}$–structures thus constructed on the grounds of the product Riemannian spaces (7.2) are y-global of the class C^1 (and not of the class C^2) on the slit tangent bundle $TM \smallsetminus 0$, and are y-global of the class C^∞ on the b-slit tangent bundle (7.1) (assuming the input Riemannian space $\mathcal{R}_{N-1}$ and the Finsleroid–axis 1-form b to be of the class C^∞).

In the two–dimensional case, $N = 2$, we have identically $\mathcal{H}_{ij} = 0$. Therefore, the implication (3.1) $\rightarrow$ (3.2) noted in Section 3 reduces to *Landsberg type* $\rightarrow$ *Berwald type* and the representation (1.10) reduces to merely $\dot{A}_i = 0$, which says us the following: Any two–dimensional $\mathcal{FF}_g^{PD}$–space is of the Berwald type, whenever the Finsleroid charge $g(x)$ is a constant via the condition (1.6), the Riemannian case being occurred if only $g = 0$. The Landsberg scalar J is zero if and only if the Finsleroid charge $g(x)$ is a constant. The two–dimensional analog to the Cartan tensor structure (2.27) is the representation $A_{ijk} = I(x)A_iA_jA_k$ with the main scalar $I(x) = |g(x)|$ which is the function of position x only. The components of A_i are singular at only $q = 0$ (the formula (A.7) in Appendix A is applicable to the two–dimensional case), whence the quantity $I(x)$ is meaningful on all the indicatrix $\mathcal{I}_x \subset T_xM$ except for the top and down points. At first sight, a suspicion may arise that we contradict to the known and remarkable theorem (see pp. 278 and 279 in the book [7]) which claims that the independence of the main scalar of tangent vectors entails strictly the Riemannian case, that is, $I = I(x)$ entails $I = 0$. However, an attentive consideration of the conditions under which the theorem has been settled out just reveals the fact that the conditions imply the y-global smoothness of at least class C^3 to be valid for the Finsler space. It is the lack of the y-global property of the class C^3 that is the reason why the two–dimensional $\mathcal{FF}_g^{PD}$–space is spared the fate of degenerating to become Riemannian and occurs being of the Berwald type proper. All the formulae obtained in Section 4 for the hh-curvature tensor are applicable at N=2. In particular, the formula (4.6) shows that Ric at $N = 2$ is not zero.

Comparison with the Randers space brings to evidence many interesting interrelations. Let the dimension N be more than 2. It should be noted first of all that the nature of the Berwald case can easily be clarified in the $\mathcal{FF}_g^{PD}$–space as well as in the Randers space, resulting in quite a similar condition for the characteristic 1-form $b = b_k(x)y^k$ to be parallel with respect to the associated Riemannian metric, $\nabla_i b_j = 0$. In both the $\mathcal{FF}_g^{PD}$–space and the Randers space, the weakly Berwald condition $A_{j|i} = 0$ entails the complete Berwald condition $A_{jmn|i} = 0$. There is, however, a significant distinction in the

structures of the Randers space and the $\mathcal{FF}_g^{PD}$–space in respect of the Landsberg–type properties. Namely, the identical vanishing $A^j \dot{A}_j = 0$ may be considered to be the weakest necessary condition to underline the full Landsberg condition $\dot{A}_{ikj} = 0$. It proves that the identical vanishing of the contraction $A^j \dot{A}_j$ is *always* valid in the $\mathcal{FF}_g^{PD}$–space with a constant Finsleroid charge and *never* in the non–Riemannian Randers space. The latter fact says us that the Randers space can *never* be of the properly Landsberg type, that is, for the Randers space the notions of being Lansberg and being Berwald are equivalent. Can the condition $\nabla_i b_j = k(x)(a_{ij} - b_i b_j)$ generate a non–Berwaldian Landsberg space? The true answer, namely "yes" in the $\mathcal{FF}_g^{PD}$–space *versus* "no" in the Randers space, can be motivated in simple words. Namely, in both spaces the dotted vector $\dot{A}_i$ under the condition $\nabla_i b_j = k(x)(a_{ij} - b_i b_j)$, as well as the very field A_i, is spanned by a linear combination of two vectors, y_i and b_i, so that such two scalars c_1, c_2 must exist that $\dot{A}_i = c_1 y_i + c_2 A_i$. Since $y^i \dot{A}_i$ is zero in any Finsler space, we must put $c_1 = 0$. The watershed is that in $\mathcal{FF}_g^{PD}$–space, – and not in the Randers space, – we also have zero for the contraction $A^i \dot{A}_i$ which entails that c_2 is also zero. So in $\mathcal{FF}_g^{PD}$–space we can safely pose the condition $\nabla_i b_j = k(x)(a_{ij} - b_i b_j)$ to get a non–Berwaldian Landsberg type, while such a door is closed for ever in the Randers space. Obviously, the distinction is rooted in difference of the algebraic structures of the Cartan tensors associated with the $\mathcal{FF}_g^{PD}$–space and with the Randers space. It is also interesting to note for comparison that the condition of the type $\nabla_i b_j = \frac{1}{2}\sigma(a_{ij} - b_i b_j)$ with $\sigma = const$ appears when characterizing the Randers spaces of constant flag curvature (see [10,11]).

Appendix A: Representations for Distinguished $\mathcal{FF}_g^{PD}$–Objects

With the notation $u_k = y^n a_{nk}$, the direct calculations of the covariant tangent vector and Finslerian metric tensor on the basis of the Finsleroid metric function K given by (2.14) yield

$$y_i = (a_{ij}y^j + gqb_i)\frac{K^2}{B} \tag{A.1}$$

and

$$g_{ij} = \left[a_{ij} + \frac{g}{B}\left((gq^2 - \frac{bS^2}{q})b_i b_j - \frac{b}{q}u_i u_j + \frac{S^2}{q}(b_i u_j + b_j u_i)\right)\right]\frac{K^2}{B}. \tag{A.2}$$

The reciprocal components $(g^{ij}) = (g_{ij})^{-1}$ read

$$g^{ij} = \left[a^{ij} + \frac{g}{q}(bb^i b^j - b^i y^j - b^j y^i) + \frac{g}{Bq}(b + gq)y^i y^j\right]\frac{B}{K^2}. \tag{A.3}$$

We also obtain

$$y_i b^i = (b + gq)\frac{K^2}{B}, \qquad g_{ij}b^j = \left(b_i + gq\frac{y_i}{K^2}\right)\frac{K^2}{B}, \tag{A.4}$$

and

$$h_{ij}b^j = \left(b_i - b\frac{y_i}{K^2}\right)\frac{K^2}{B}. \tag{A.5}$$

The determinant of the tensor (A.2) is everywhere positive:

$$\det(g_{ij}) = \left(\frac{K^2}{B}\right)^N \det(a_{ij}) > 0. \tag{A.6}$$

By the help of the formulas (A.2)–(A.4) we find

$$A_i = \frac{NK}{2}g\frac{1}{q}(b_i - \frac{b}{K^2}y_i) \tag{A.7}$$

and

$$A^i = \frac{N}{2}g\frac{1}{qK}\left[Bb^i - (b + gq)y^i\right] \tag{A.8}$$

together with

$$A_ib^i = \frac{N}{2}gq\frac{K}{B}, \qquad A^ib_i = \frac{N}{2}gq\frac{1}{K}. \tag{A.9}$$

These formulas are convenient to verify the algebraic structure (2.27) and the contraction (2.28). Also,

$$A_{ij} := K\partial A_i/\partial y^j + l_iA_j = -\frac{N}{2}\frac{gb}{q}\mathcal{H}_{ij} + \frac{2}{N}A_iA_j \tag{A.10}$$

with

$$\mathcal{H}_{ij} = h_{ij} - \frac{A_iA_j}{A_nA^n}. \tag{A.11}$$

The last tensor fulfills obviously the identities

$$\mathcal{H}_{ij}y^j = 0, \qquad \mathcal{H}_{ij}A^j = 0, \tag{A.12}$$

which in turn entails

$$\mathcal{H}_{ij}b^j = b_j\mathcal{H}_i{}^j = 0 \tag{A.13}$$

because A^i are linear combinations of y^i and b^i (see (A.8)). We also have

$$g^{ij}\mathcal{H}_{ij} = N - 2, \tag{A.14}$$

$$g^{mn}\mathcal{H}_{im}\mathcal{H}_{jn} = \mathcal{H}_{ij}, \tag{A.15}$$

$$\mathcal{H}_i{}^j = \delta_i{}^j - b_ib^j - \frac{1}{q^2}(a_{in}y^n - bb_i)(y^j - bb^j) \equiv r_i{}^j - \frac{1}{q^2}v_iv^j, \tag{A.16}$$

and

$$K\left(\frac{\partial\mathcal{H}_{ij}}{\partial y^k} - \frac{\partial\mathcal{H}_{kj}}{\partial y^i}\right) = l_k\mathcal{H}_{ij} - l_i\mathcal{H}_{kj} - \frac{1}{A_nA^n}\frac{Ngb}{2q}(A_k\mathcal{H}_{ij} - A_i\mathcal{H}_{kj}) \tag{A.17}$$

together with

$$\frac{\partial(q^2\mathcal{H}_i^j)}{\partial y^k} - \frac{\partial(q^2\mathcal{H}_k^j)}{\partial y^i} = 3\left[(\delta_i{}^j - b_ib^j)(a_{kn}y^n - bb_k) - (\delta_k{}^j - b_kb^j)(a_{in}y^n - bb_i)\right]$$

$$= 3\left[\mathcal{H}_i^j(a_{kn}y^n - bb_k) - \mathcal{H}_k^j(a_{in}y^n - bb_i)\right]. \tag{A.18}$$

The structure (2.27) of the $\mathcal{FF}_g^{PD}$–space Cartan tensor is such that

$$A_kA_i{}^k{}_j = \frac{1}{N}(A_iA_j + h_{ij}A_kA^k) = \frac{1}{N}(2A_iA_j + \mathcal{H}_{ij}A_kA^k), \tag{A.19}$$

so that the tensor

$$\tau_{ij} := A_{ij} - A_k A_i{}^k{}_j \tag{A.20}$$

is equal to

$$\tau_{ij} = -\frac{N}{4}\frac{g(2b+gq)}{q}\mathcal{H}_{ij}. \tag{A.21}$$

By comparing (A.21) with (A.12) and (A.13), the identities $\tau_{ij}y^j = 0$, $\tau_{ij}A^j = 0$, and

$$\tau_{ij}b^j = b_j\tau_i{}^j = 0 \tag{A.22}$$

just follow.

The tensor

$$\tau_{ijmn} := K\partial A_{jmn}/\partial y^i - A_{ij}{}^h A_{hmn} - A_{im}{}^h A_{hjn} - A_{in}{}^h A_{hjm} + l_j A_{imn} + l_m A_{ijn} + l_n A_{ijm} \tag{A.23}$$

can be expressed as follows:

$$\tau_{ijmn} = -\frac{g(2b+gq)}{4q}(\mathcal{H}_{ij}\mathcal{H}_{mn} + \mathcal{H}_{im}\mathcal{H}_{jn} + \mathcal{H}_{in}\mathcal{H}_{jm}), \tag{A.24}$$

showing the total symmetry in all four indices and the properties

$$\tau_{ij} = g^{mn}\tau_{ijmn}$$

and

$$y^i\tau_{ijmn} = 0, \qquad A^i\tau_{ijmn} = 0, \qquad b^i\tau_{ijmn} = 0. \tag{A.25}$$

We use the Riemannian covariant derivative

$$\nabla_i b_j := \partial_i b_j - b_k a^k{}_{ij}, \tag{A.26}$$

where

$$a^k{}_{ij} := \frac{1}{2}a^{kn}(\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji}) \tag{A.27}$$

are the Christoffel symbols given rise to by the associated Riemannian metric $\mathcal{S}$.

For the hh-curvature tensor $R^i{}_k$ we use the formula

$$K^2 R^i{}_k := 2\frac{\partial \bar{G}^i}{\partial x^k} - \frac{\partial \bar{G}^i}{\partial y^j}\frac{\partial \bar{G}^j}{\partial y^k} - y^j\frac{\partial^2 \bar{G}^i}{\partial x^j \partial y^k} + 2\bar{G}^j\frac{\partial^2 \bar{G}^i}{\partial y^k \partial y^j} \tag{A.28}$$

(which is tantamount to the definition (3.8.7) on p. 66 of the book [7]; $\bar{G}^i = \frac{1}{2}\gamma^i{}_{nm}y^n y^m$, with the Finslerian Christoffel symbols $\gamma^i{}_{nm}$). The concomitant tensors

$$R^i{}_{km} := \frac{1}{3K}\left(\frac{\partial(K^2 R^i{}_k)}{\partial y^m} - \frac{\partial(K^2 R^i{}_m)}{\partial y^k}\right) \tag{A.29}$$

and

$$R_n{}^i{}_{km} := \frac{\partial(K R^i{}_{km})}{\partial y^n} \tag{A.30}$$

arise.

The cyclic identity

$$R_j{}^i{}_{kl|t} + R_j{}^i{}_{lt|k} + R_j{}^i{}_{tk|l} = P_j{}^i{}_{ku}R^u{}_{lt} + P_j{}^i{}_{lu}R^u{}_{tk} + P_j{}^i{}_{tu}R^u{}_{kl} \tag{A.31}$$

(the formula (3.5.3) on p. 58 of the book [7]) is valid in any Finsler space. If we contract (A.31) by $g^{jl}\delta^k{}_i$, we get:

$$g^{jl}\left(R_j{}^i{}_{il|t} + R_j{}^i{}_{lt|i} + R_j{}^i{}_{ti|l}\right) = P^{li}{}_{iu}R^u{}_{lt} + P^{li}{}_{lu}R^u{}_{ti} + P^{li}{}_{tu}R^u{}_{il}. \qquad (A.32)$$

Under the Landsberg condition, the tensor P_{ijkl} is symmetric in all of its four indices and, therefore, the previous identity reduces to merely

$$g^{jl}\left(R_j{}^i{}_{il|t} + R_j{}^i{}_{lt|i} + R_j{}^i{}_{ti|l}\right) = 0, \qquad (A.33)$$

which can also be written as

$$\rho^i{}_{j|i} \equiv 0 \qquad (A.34)$$

with

$$\rho_{ij} := \frac{1}{2}(R_i{}^m{}_{mj} + R^m{}_{ijm}) - \frac{1}{2}g_{ij}R^{mn}{}_{nm}. \qquad (A.35)$$

This conservation law (A.34)–(A.35) is valid in any Finsler space of the Landsberg type and, therefore, in any Landsberg case of the $\mathcal{FF}_g^{PD}$–space studied.

To verify (1.31)–(1.32) the reader can use the relationship

$$\dot{A}_{jkl} = -\frac{1}{4}y_i\frac{\partial^3\gamma^i{}_{nm}y^ny^m}{\partial y^j\partial y^k\partial y^l} \qquad (A.36)$$

(the formula (3.8.5) on p. 65 of the book [7]) which is valid in any Finsler space. It is easy to check that the insertion of the particular coefficients (1.24) in (A.36) results in the Landsbergian $\dot{A}_{jkl} = 0$.

Finally, we are able to compute components of the hv-curvature tensor $P_k{}^i{}_{mn}$. To this end we use the formula

$$P_k{}^i{}_{mn} = K\frac{\partial\left(\dot{A}^i{}_{km} - \frac{1}{2}G^i{}_{km}\right)}{\partial y^n} \qquad (A.37)$$

(see (3.8.4) on p. 65 of [7]) which is valid in any Finsler space. In the $\mathcal{FF}_g^{PD}$–space under the only condition that the Finsleroid charge is a constant, direct tedious calculations yield the result explicitly in terms of the lengthened angular metric tensor $\mathcal{H}_{mn}$, namely

$$\frac{2}{K}P_k{}^i{}_{mn} = \frac{g}{q}\left(f^i{}_k\mu_{mn} + f^i{}_m\mu_{kn} - (\nabla_kb_m + \nabla_mb_k)\mu^i{}_n\right)$$

$$+\frac{g}{q^3}\left(t_k\mu^i{}_{mn} - q^2(\nabla_kb_n)\mu^i{}_m + t_m\mu^i{}_{kn} - q^2(\nabla_mb_n)\mu^i{}_k - t^i\mu_{kmn} + q^2(\nabla^ib_n)\mu_{km}\right)$$

$$-\frac{g^2}{2q^2}\left[\left(b^j\nabla_jb^i - \frac{1}{q^2}v^iy^hb^j\nabla_jb_h\right)\mu_{kmn} - \left(v_nb^j\nabla_jb^i - v^ib^j\nabla_jb_n\right)\mu_{km}\right]$$

$$-\frac{g^2}{2q^2}\left[\left(b^j\nabla_jb_k - \frac{1}{q^2}v_ky^hb^j\nabla_jb_h\right)\mu^i{}_{mn} - \left(v_nb^j\nabla_jb_k - v_kb^j\nabla_jb_n\right)\mu^i{}_m\right]$$

$$-\frac{g^2}{2q^2}\left[\left(b^j\nabla_j b_m - \frac{1}{q^2}v_m y^h b^j\nabla_j b_h\right)\mu^i_{kn} - \left(v_n b^j\nabla_j b_m - v_m b^j\nabla_j b_n\right)\mu^i_k\right]$$

$$-\frac{g^2}{2q^2}y^h b^j\nabla_j b_h\left(\mu^i_n\mu_{km} + \mu^i_m\mu_{kn} + \mu^i_k\mu_{mn}\right),\tag{A.38}$$

where

$$\mu^i_n = r^i_n - \frac{1}{q^2}v^i v_n \equiv \mathcal{H}^i_n, \qquad \mu_{in} = r_{in} - \frac{1}{q^2}v_i v_n \equiv \mu^j_n a_{ji} \equiv \mathcal{H}_{in}\frac{B}{K^2},\tag{A.39}$$

$$\mu^i_{kn} = r^i_k v_n + r^i_n v_k + r_{nk}v^i - \frac{3}{q^2}v^i v_n v_k \equiv \mathcal{H}^i_k v_n + \mathcal{H}^i_n v_k + \mathcal{H}_{nk}\frac{B}{K^2}v^i,\tag{A.40}$$

$$\mu_{kmn} = r_{km}v_n + r_{kn}v_m + r_{mn}v_k - \frac{3}{q^2}v_k v_m v_n \equiv \mu^j_{mn}a_{jk}$$

$$\equiv \left(\mathcal{H}_{km}v_n + \mathcal{H}_{kn}v_m + \mathcal{H}_{mn}v_k\right)\frac{B}{K^2},\tag{A.41}$$

and

$$t_k = y^h\nabla_k b_h, \qquad t^i = a^{ik}t_k.\tag{A.42}$$

It is easy to observe that

$$P_k^i{}_{mn}y^n = 0, \qquad P_k^i{}_{mn} = P_m^i{}_{kn},\tag{A.43}$$

and

$$y^k P_k^i{}_{mn} = -\dot{A}^i{}_{mn},\tag{A.44}$$

where the right–hand side agrees exactly with the right–hand side of (1.31). It is also easy to verify that upon the Landsbergian condition (1.16) the above tensor $P_k^i{}_{mn}$ given by (A.38) reduces to the tensor (1.18).

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